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Simulation of Conditioned Diffusions on the Flat Torus

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Summary

- There exists an ubundance of statistical methods for analysing raw data residing in Euclidean space, however, most of which does not apply when the underlying data space is more complicated than the standard Euclidean space.
- Some of the statistical methods relies on simulation of Brownian bridges, which are efficient tools for statistical inference, e.g. Kolmogorov-Smirnov's goodness of fit test (which quantifies a distance between an empirical distribution function and a reference distribution function).
- We propose a method for simulating diffusion bridges from a to b on the flat torus over the interval $[0, T]$, which approximate the true Brownian bridge from a to b over $[0, T]$. This is done by considering a 2-dimensional Euclidean bridge process, which is then projected onto the flat torus. (see Fig. 1)

The Geometrical Idea

- The geometrical intuition of the flat **torus** is easy to comprehend; Take a piece of elastic paper and identify top and bottom so that it forms a cylinder, then by stretching the cylinder the two ends can meet to form a donut shape. This is illustrated in Fig. 1.
- Let $\mathbb{T}^2$ denote the 2-dimensional torus obtained by the algebraic operation $\mathbb{R}^2/\mathbb{Z}^2$, i.e. we can write it as the set

$$\mathbb{T}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : -\tfrac{1}{2} \le x_i < \tfrac{1}{2}, i = 1, 2\}.$$

- The flat torus inherits the Euclidean structure, hence the word **flat** and the map π defines a canonical surjection (see Fig. 1).

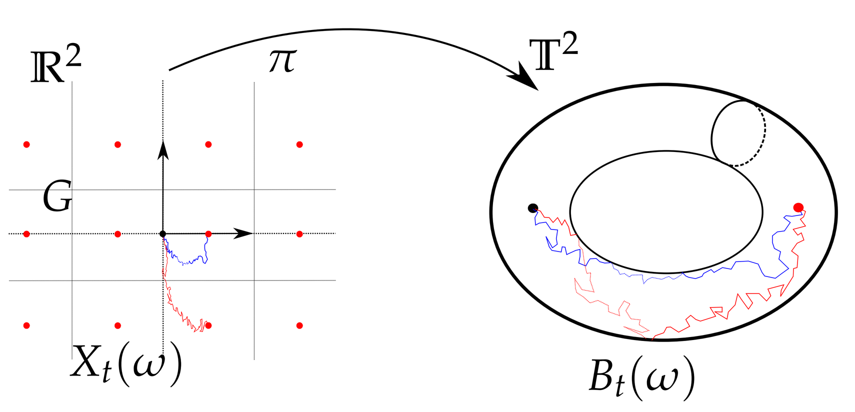


Figure 1: The figure illustrates the possibility of the diffusion path going an arbitrary number of times around the torus, starting at the black dot and ending in the red. This is illustrated by the red path. The conditioning on single point in $\mathbb{T}^2$ therefore leads to conditioning on multiple points in $\mathbb{R}^2$. Left: Two paths from the same two-dimensional process with multiple endpoints. Right: The projection of the two paths onto the torus.

- For a point $b \in \mathbb{T}^2$ (red point on the torus in Fig. 1) the pullback $\pi^{-1}(b)$ gives a collection of points in $\mathbb{R}^2$ (red points in $\mathbb{R}^2$ in Fig. 1). The grid G in Fig. 1 denotes the set of points where there does not exists a unique shortest path to a point in $\pi^{-1}(b)$.

Brownian Bridges

- A standard **Brownian motion** $W = (W_t)_{t \ge 0}$ is an almost surely continuous stochastic process with independent increments satisfying

$$W_0 = 0 \quad a.s., \quad W_t - W_s \sim N(0, t-s), \quad for \quad t > s.$$

- A **Brownian bridge** $B = (B_t)_{0 \le t \le T}$ from a to b at time T, is a conditional Brownian motion

$$B = W | (W_0 = a, W_T = b) \tag{1}$$

- Following from Doob's h-transform it can be written as the stochastic differential equation (SDE)

$$dB_t = \nabla_x \log(p(t, x; T, b))\big|_{x=B_t} dt + dW_t,$$

where in the standard Euclidean setup, the transition density will have the form

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{||x-y||^2}{2(t-s)}\right), \quad s < t.$$

- The drift term acts as the guiding term pulling the process towards the end point (see e.g. Fig. 2a for a visualisation of this).
- In the particular case of the flat torus, the transition density will be a sum over $y \in \pi^{-1}(b)$.

Diffusion Bridges on the Flat Torus

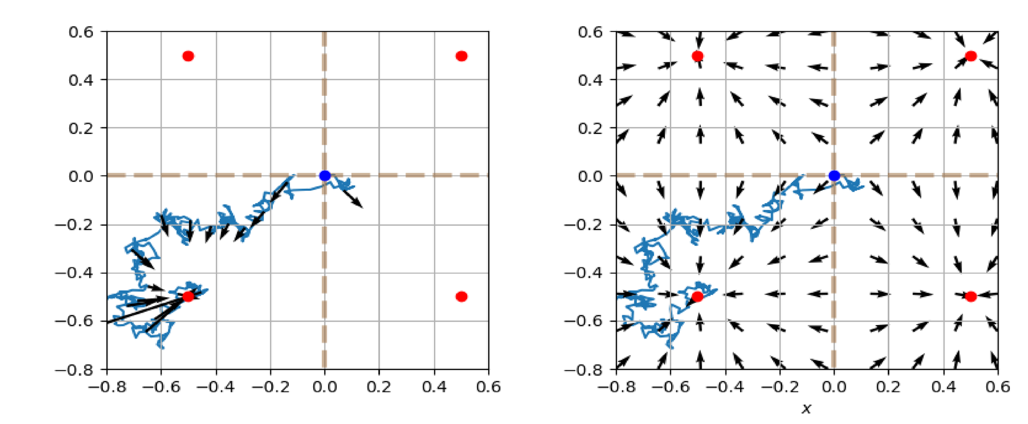
- We consider the diffusion process on $[0, T)$, for some fixed positive T, defined by

$$dX_t = 1_{G^c}(X_t)\frac{\alpha(X_t) - X_t}{T - t}dt + \sigma dW_t, \qquad X_0 = x_0 \tag{2}$$

where $\sigma > 0$ and α is defined by

$$\alpha(X_t) = \underset{y \in \pi^{-1}(b)}{\arg\min} \|y - X_t\|,$$

- The solution of (2) guides to the nearest point in $\pi^{-1}(b)$ which is illustrated in Fig. 2. In particular, the drift term is increasing as $t \to T$ (see Fig. 2a and Fig. 3b).



(a) Drift term (b) Vector field

Figure 2: Figure 2a depicts the evolution of the drift term. It shows how the pull from the drift becomes stronger near the end. Figure 2b shows the underlying vector field.

Proposition 1 *There exist a strong solution of* (2) *on* $[0, T)$*, which is strongly unique.*

Theorem 1 *The law of* (2) *is equivalent to the law of the true Brownian bridge on* $[0, T)$.

- Paths from the proposed model are plotted against paths from the true Brownian bridge on the flat torus in Fig. 4.

Numerical Implementations

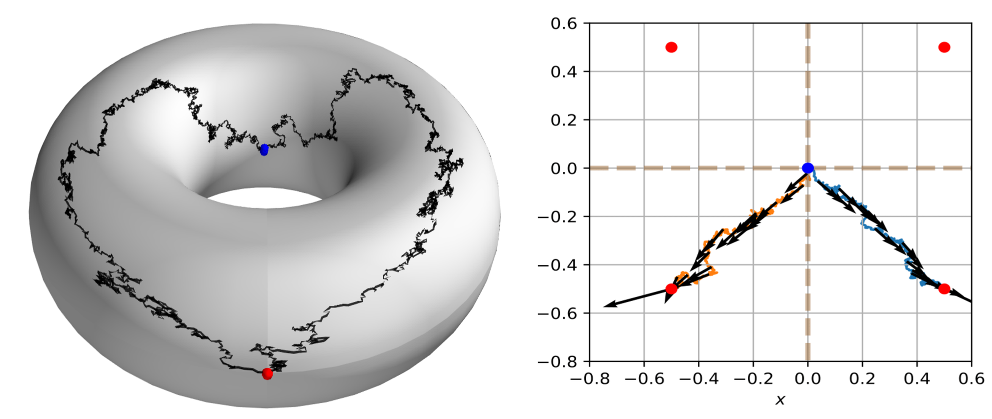
- For the numerical implementation of the proposed SDE in equation (2) we implemented the **Euler-Maruyama scheme**.
- Step 1: Take $n+1$ equidistant discretization points of the time interval $0 = t_0 < t_1 < ... < t_n = T$, with $t_{i+1} - t_i = \Delta t$, then

$$\Delta W_{t_{i+1}} = W_{t_{i+1}} - W_{t_i} \sim N(0, \Delta t)$$

where $(W_t)_{t \ge 0}$ is a standard Brownian motion.

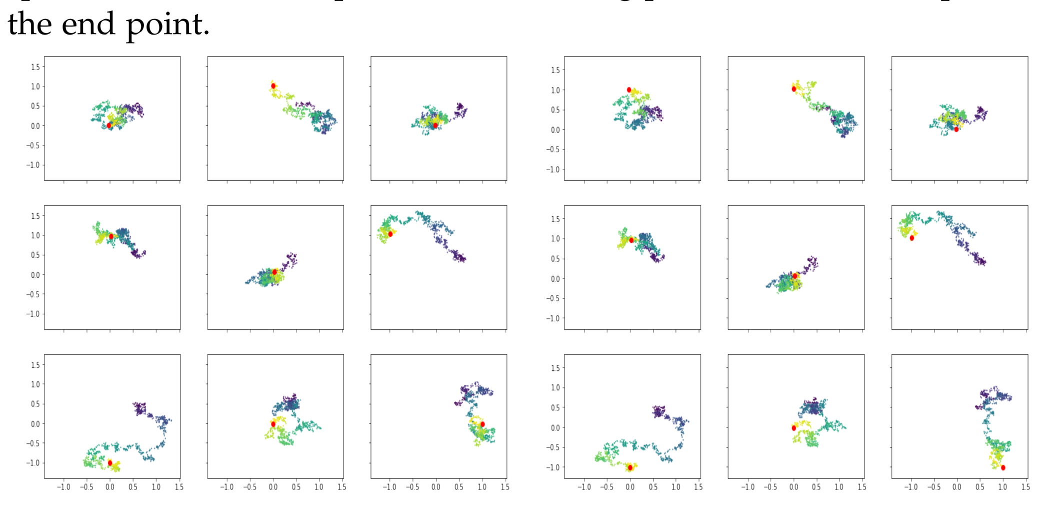
- Step 2: Set $x_0 = a \in \mathbb{R}^2$ and define iteratively

$$x_{t_{i+1}} = x_{t_i} + \frac{\alpha(x_{t_i}) - x_{t_i}}{T - t_i}\Delta t + \sigma \Delta W_{t_i}.$$



(a) Paths visualized on an embedded torus. (b) The two Euclidean paths that are mapped onto the torus.

Figure 3: Two different paths visualized both on the torus and in Eucliden space. The blue dot represents the starting point and the red represents the end point.



(a) 9 realisations of the proposed bridge process. (b) 9 realisations from the true bridge process.

Figure 4: The color of the paths indicate the time evolution and the red dots their end points, respectively.

Forthcoming Research

- We aim to extend the above results to the case of more general manifolds e.g. spheres.

Acknowledgements

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